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# 極大平面グラフの 独立全域木を求める線形時間アルゴリズム

## A Linear-Time Algorithm to Find Independent Spanning Trees in Maximal Planar Graphs

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**Abstract:** Given a graph $G$, a designated vertex $r$ and a natural number $k$, we wish to find $k$ " independent " spanning trees of $G$ rooted at $r$, that is, $k$ spanning trees such that, for any vertex $v$, the $k$ paths connecting $r$ and $v$ in the $k$ trees are internally disjoint in $G$. In this paper we give a linear-time algorithm to find $k$ independent spanning trees in a $k$-connected maximal planar graph rooted at any vertex.

**Key ward:** graph, algorithm, independent spanning trees

## 1 Introduction

Given a graph $G = (V, E)$, a designated vertex $r \in V$ and a natural number $k$, we wish to find $k$ spanning trees $T_1, T_2, \cdots, T_k$ of $G$ such that, for any vertex $v$, the $k$ paths connecting $r$ and $v$ in $T_1, T_2, \cdots, T_k$ are internally disjoint in $G$, that is, any two of them have no common intermediate vertices. Such $k$ trees are called *$k$ independent spanning trees of $G$ rooted at $r$*. Five independent spanning trees are drawn in Fig. 1 by thick lines. Independent spanning trees have applications to fault-tolerant protocols in networks [BI96, DHSS84, IR88, OIBI96].

Given a graph $G = (V, E)$ of $n$ vertices and $m$ edges, and a designated vertex $r \in V$, one can find two independent spanning trees of $G$ rooted at any vertex in linear time if $G$ is biconnected [BTV96, BTV99, IR88], and find three independent spanning trees of $G$ rooted at any vertex in $O(mn)$ and $O(n^2)$ time if $G$ is triconnected [BTV96, BTV99, CM88]. It is conjectured that, for any $k \geq 1$, every $k$-connected graph has $k$ independent spanning trees rooted at any vertex [KS92, ZI89]. For general graphs with $k \geq 4$ the conjecture is still open, however, for planar graphs the conjecture is verified by Huck for $k = 4$ [H94] and $k = 5$ [H99] (i.e., for all planar graphs, since every planar graph has a vertex of degree at most 5 [W96, p269] means there is no 6-connected planar graph). The proof in [H99] yields an algorithm to actually find $k$ independent spanning trees in a $k$-connected planar graph, but it takes time $O(n^3)$. On the other hand, for $k$-connected maximal planar graphs we can find $k$ independent spanning trees in linear time for $k = 2$ [BTV96, BTV99, IR88], $k = 3$ [BTV96, BTV99, S90] and $k = 4$ [MTNN98].

In this paper we give a simple linear-time algorithm to find five independent spanning trees of a 5-connected maximal planar graph rooted at any designated vertex. Note that, since there

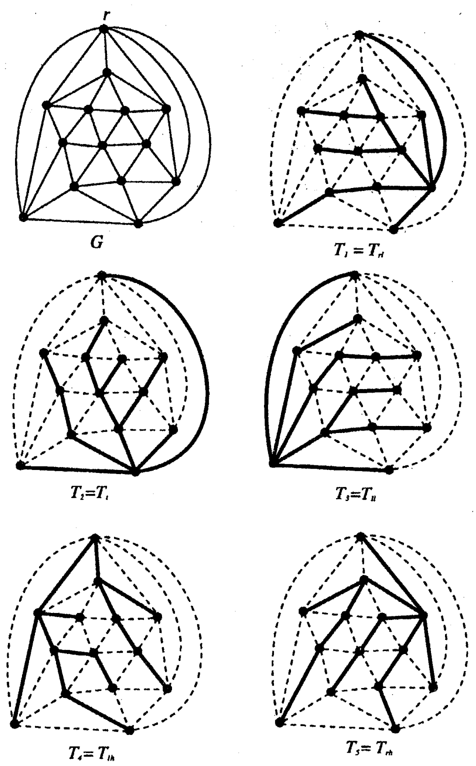


図 1: Five independent spanning trees $T_1, T_2, T_3, T_4$ and $T_5$ of a graph $G$ rooted at $r$.

is no 6-connected planar graph, our result, together with previous results [BTV96, BTV99, IR88, MTNN98, S90], yields a linear-time algorithm to find $k$ independent spanning trees in a $k$-connected maximal planar graph rooted at any designated vertex. Our algorithm is based on a " 5-canonical decomposition" of a 5-connected maximal planar graph, which is a generalization of an *st*-numbering [E79], a canonical ordering [K96], a canonical decomposition [CK93, CK97], a canonical 4-ordering [KH94] and a 4-canonical decomposition [MTNN98, NRN97].

The remainder of the paper is organized as follows. In Section 2 we introduce some definitions. In Section 3 we present our algorithm to find five independent spanning trees based on a 5-canonical decomposition. In Section 4 we give an algorithm to find a 5-canonical decomposition. Finally we put conclusion in Section 5.

## 2 Preliminaries

In this section we introduce some definitions.

Let $G = (V, E)$ be a connected graph with vertex set $V$ and edge set $E$. Throughout the paper we denote by $n$ the number of vertices in $G$, and we always assume that $n > 5$. An edge joining vertices $u$ and $v$ is denoted by $(u, v)$. The *degree* of a vertex $v$ in $G$, denoted by $d(v, G)$ or simply by $d(v)$, is the number of neighbors of $v$ in $G$. The *connectivity* $\kappa(G)$ of a graph $G$ is the minimum number of vertices whose removal results in a disconnected graph or a single-vertex graph $K_1$. A graph $G$ is *k-connected* if $\kappa(G) \geq k$. A *path* in a graph is an ordered list of distinct vertices $v_1, v_2, \cdots, v_l$ such that $v_{i-1}v_i$ is an edge for all $i$, $2 \leq i \leq l$. We say that two paths having common start and end vertices are *internally disjoint* if their intermediate vertices are disjoint. We also say that a set of paths having common start and end vertices are *internally disjoint* if every pair of paths in the set are internally disjoint.

A graph is *planar* if it can be embedded in the plane so that no two edges intersect geometrically except at a vertex to which they are both incident. A planar graph $G$ is *maximal* if all faces including the outer face are triangles in some planar embedding of $G$. Essentially each maximal planar graph has a unique planar embedding except for the choice of the outer face. A *plane graph* is a planar graph with a fixed planar embedding. The *contour* $C_o(G)$ of a biconnected plane graph $G$ is the clockwise (simple) cycle on the outer face. We write $C_o(G) = (w_1, w_2, \cdots, w_h)$ if the vertices $w_1, w_2, \cdots, w_h$ on $C_o(G)$ appear in this order.

## 3 Algorithm

In this section we give our algorithm to find five independent spanning trees of a 5-connected maximal planar graph rooted at any designated vertex.

Given a 5-connected maximal planar graph $G = (V, E)$ and a designated vertex $r \in V$, we first find a planar embedding of $G$ in which $r$ is located on $C_o(G)$. Let $G' = G - \{r\}$ be the plane subgraph of $G$ induced by $V - \{r\}$. In Fig. 2 (a) $G$ is drawn by solid and dotted lines, and $G'$ by solid lines. Since $G$ is 5-connected, $d(r) \geq 5$. We may assume that all the neighbors $r_1, r_2, \cdots, r_{d(r)}$ of $r$ in $G$ appear on $C_o(G')$ clockwise in this order. Now $C_o(G') = (r_1, r_2, \cdots, r_{d(r)})$. We add to $G'$ two new vertices $r_b$ and $r_t$, join $r_b$ with $r_1$,$r_2$ and $r_3$, and join $r_t$ with $r_4, r_5, \cdots, r_{d(r)}$. Let $G''$ be the resulting plane graph, where vertices $r_1, r_b, r_3, r_4, r_t$ and $r_{d(r)}$ appear on $C_o(G'')$ clockwise in this order. Fig. 2 (b) illustrates $G''$.

Let $\Pi = (W_1, W_2, \cdots, W_m)$ be a partition of the vertex set $V - \{r\}$ of $G'$. We denote by $G_k$, $1 \leq k \leq m$, the plane subgraph of $G''$ induced by $\{r_b\} \bigcup W_1 \bigcup W_2 \bigcup \cdots \bigcup W_k$. We denote by $\overline{G_k}$, $0 \leq k \leq m-1$, the plane subgraph of $G''$ induced by $W_{k+1} \bigcup W_{k+2} \bigcup \cdots \bigcup W_m \bigcup \{r_t\}$. We assume that if $1 < k \leq m$ and $W_k = \{u_1, u_2, \cdots, u_l\}$ then vertices $u_1, u_2, \cdots, u_l$ consecutively appear on $C_o(G_k)$ clockwise in this order. Note that for $k = 1$ we don't assume such a condition. A partition $\Pi = (W_1, W_2, \cdots, W_m)$ of $V - \{r\}$ is called a *5-canonical decomposition* of $G'$ if the following three conditions (co1)–(co3) are satisfied.

(co1) $W_1 = \{r_1, r_2, r_3\} \bigcup \{u_2, u_3, \cdots, u_{d(r_2)-2}\}$, where vertices $u_2, u_3, \cdots, u_{d(r_2)-2}$ are the neighbors of $r_2$ except $r_1, r_3, r_b$, and $W_m = \{r_{d(r)-1}, r_{d(r)}\}$

(co2) For each $k$, $1 \leq k \leq m$, $G_k$ is triconnected, and for each $k$, $0 \leq k \leq m-1$, $\overline{G_k}$ is biconnected (See Fig. 3.); and

(co3) For each $k$, $1 < k < m$, one of the following two conditions holds (See Fig. 3. The vertices in $W_k$ are drawn in black dots):

(a) $|W_k| \geq 2$, and each vertex $u \in W_k$ satisfies $d(u, G_k) = 3$ and $d(u, \overline{G_{k-1}}) \geq 3$; and

(b) $|W_k| = 1$, and the vertex $u \in W_k$ satisfies $d(u, G_k) \geq 3$ and $d(u, \overline{G_{k-1}}) \geq 2$.

Fig. 2 (b) illustrates a 5-canonical decomposition of $G' = G - \{r\}$, where $G'$ are drawn in solid lines and each set $W_i$ is indicated by an oval drawn in a dotted line. A 5-canonical decomposition is a generalization of an "*st*-numbering" [E79], a "canonical ordering" [K96], a "canonical decomposition" [CK93, CK97], a "canonical 4-ordering" [KH94] and a "4-canonical decomposition" [MTNN98, NRN97].

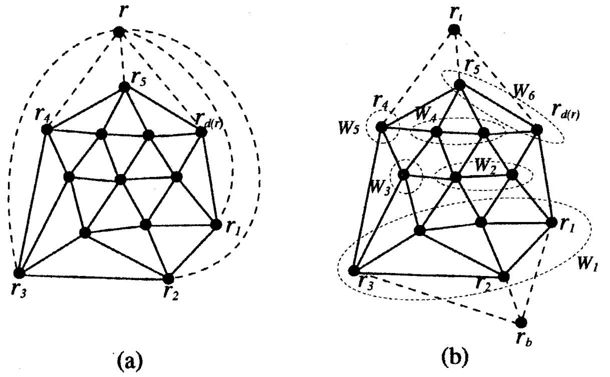


図 2: (a) Five-connected plane graph $G$ and (b) plane graph $G''$.

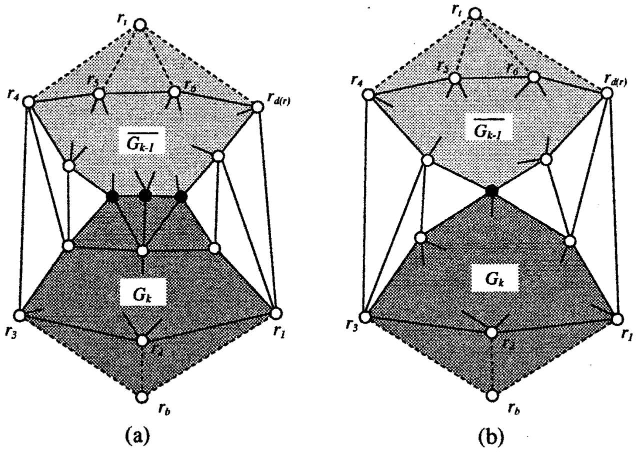


図 3: Two conditions for (co3).

We have the following lemma. We will give a proof of Lemma 3.1 in Section 4.

**Lemma 3.1** Let $G = (V, E)$ be a 5-connected maximal plane graph, and let $r$ be a designated vertex on $C_o(G)$. Then $G' = G - \{r\}$ has a 5-canonical decomposition $\Pi$. Furthermore $\Pi$ can be found in linear time.

We need a few more definitions to describe our algorithm. For a vertex $v \in V - \{r\}$ we write $N(v) = \{v_1, v_2, \cdots, v_{d(v)}\}$ if $v_1, v_2, \cdots, v_{d(v)}$ are the neighbors of vertex $v$ in $G''$ and appear around $v$ clockwise in this order. To each vertex $v \in V - \{r\}$ we assign five edges incident to $v$ in $G''$ as *the right leg* $rl(v)$, *the tail* $t(v)$, *the left leg* $ll(v)$, *the left hand* $lh(v)$ and *the right hand* $rh(v)$ as follows. We will show later that such an assignment immediately yields five independent spanning trees of $G$. Let $v \in W_k$ for some $k$, $1 \leq k \leq m$, then there are the following four cases to consider.

**Case 1**: $k = 1$. (See Fig. 4(a).)

Now $W_1 = \{r_1, r_2, r_3\} \bigcup \{u_2, u_3, \cdots, u_{d(r_2)-2}\}$. We may assume that vertices $u_2, u_3, \cdots, u_{d(r_2)-2}$ consecutively appear on $C_o(G_1)$ clockwise in this order. Let $u_1 = r_3, u_0 = r_b, u_{d(r_2)-1} = r_1$ and $u_{d(r_2)} = r_b$. For each $u_i \in W_1 - \{r_2\}$ we define $rl(u_i) = (u_i, u_{i+1})$, $t(u_i) = (u_i, r_2)$, $ll(u_i) = (u_i, u_{i-1})$, $lh(u_i) = (u_i, v_1)$, and $rh(u_i) = (u_i, v_{d(u_i)-3})$ where we assume $N(u_i) = \{u_{i-1}, v_1, v_2, \cdots, v_{d(u_i)-3}, u_{i+1}, r_2\}$. For $r_2$ we define $rl(r_2) = (r_2, r_1)$, $t(r_2) = (r_2, r_b)$, $ll(r_2) = (r_2, r_3)$, $lh(r_2) = (r_2, u_2)$, and $rh(r_2) = (r_2, u_{d(r_2)-2})$.

**Case 2**: $W_k$ satisfies Condition (a) of (co3). (See Fig. 4(b).)

Let $W_k = \{u_1, u_2, \cdots, u_l\}$. Since $d(u_i, G_k) = 3$ for each vertex $u_i$ and $G$ is maximal planar, vertices $u_1, u_2, \cdots, u_l$ have exactly one common neighbor, say $v$, in $G_k$. Let $u_0$ be the vertex on $C_o(G_k)$ preceding $u_1$, and let $u_{l+1}$ be the vertex on $C_o(G_k)$ succeeding $u_l$. For each $u_i \in W_k$ we define $rl(u_i) = (u_i, u_{i+1})$, $t(u_i) = (u_i, v)$, $ll(u_i) = (u_i, u_{i-1})$, $lh(u_i) = (u_i, v_1)$, and $rh(u_i) = (u_i, v_{d(u_i)-3})$ where we assume $N(u_i) = \{u_{i-1}, v_1, v_2, \cdots, v_{d(u_i)-3}, u_{i+1}, v\}$.

**Case 3**: $W_k$ satisfies Condition (b) of (co3). (See Fig. 4(c).)

Let $W_k = \{u\}$, let $u'$ be the vertex on $C_o(G_k)$ preceding $u$, and let $u''$ be the vertex on $C_o(G_k)$ succeeding $u$. Let $N(u) = \{u', v_1, v_2, \cdots, v_{d(u)-1}\}$, and let $u'' = v_x$ for some $x$, $3 \leq x \leq d(u) - 2$. Then $rl(u) = (u, u'')$, $t(u) = (u, v_{d(u)-1})$, $ll(u) = (u, u')$, $lh(u) = (u, v_1)$, and $rh(u) = (u, v_{x-1})$.

**Case 4**: $k = m$. (See Fig. 4(d).)

Now $W_m = \{r_{d(r)-1}, r_{d(r)}\}$. Let $u_0 = r_t$, $u_1 = r_{d(r)-1}$, $u_2 = r_{d(r)}$ and $u_3 = r_t$. For each $u_i \in W_k$ we define $rl(u_i) = (u_i, v_1)$, $t(u_i) = (u_i, v_{d(u_i)-3})$, $ll(u_i) = (u_i, v_{d(u_i)-2})$, $lh(u_i) = (u_i, u_{i-1})$, and $rh(u_i) = (u_i, u_{i+1})$ where we assume $N(u_i) = \{u_{i+1}, v_1, v_2, \cdots, v_{d(u_i)-2}, u_{i-1}\}$.

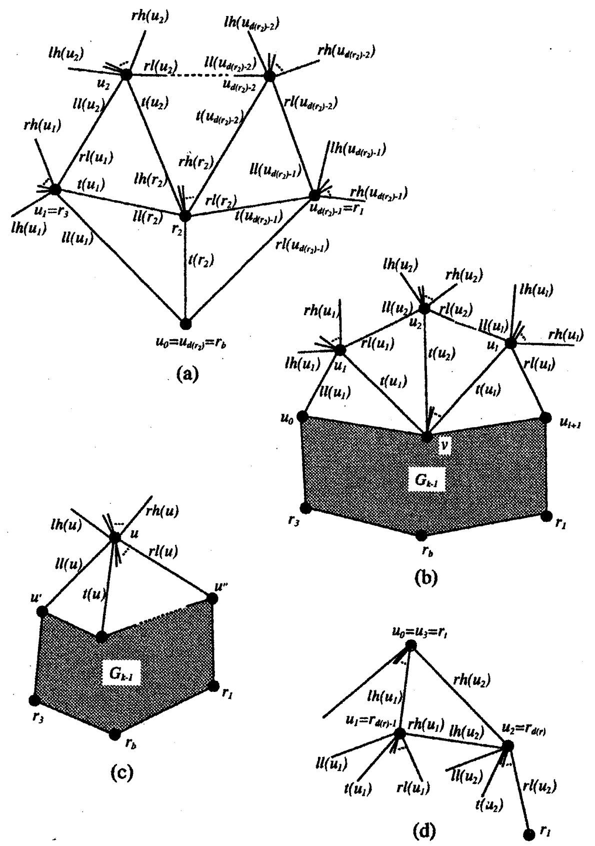

図 4: Assignment.

We are now ready to give our algorithm.

**Procedure FiveTrees($G, r$)**
**begin**

1 Find a planar embedding of $G$ such that $r \in C_o(G)$;

2 Find a 5-canonical decomposition $\Pi = (W_1, W_2, \cdots, W_m)$ of $G - \{r\}$;

3 For each vertex $v \in V - \{r\}$ find $rl(v), t(v), ll(v), lh(v)$ and $rh(v)$;

4 Let $T_{rl}$ be a graph induced by the right legs of all vertices in $V - \{r\}$;

5 Let $T_t$ be a graph induced by the tails of all vertices in $V - \{r\}$;

6 Let $T_{ll}$ be a graph induced by the left legs of all vertices in $V - \{r\}$;

7 Let $T_{lh}$ be a graph induced by the left hands of all vertices in $V - \{r\}$;

8 Let $T_{rh}$ be a graph induced by the right hands of all vertices in $V - \{r\}$;

9 Regard vertex $r_b$ in trees $T_{rl}$, $T_t$ and $T_{ll}$ as vertex $r$;

10 Regard vertex $r_t$ in trees $T_{lh}$ and $T_{rh}$ as vertex $r$;

11 **return** $T_{rl}, T_t, T_{ll}, T_{lh}$ and $T_{rh}$ as five independent spanning trees of $G$.

**end**

We then verify the correctness of our algorithm. Assume that $G = (V, E)$ is a 5-connected maximal planar graph with a designated vertex $r \in V$, and that Algorithm FiveTrees finds a 5-canonical decomposition $\Pi = (W_1, W_2, \cdots, W_m)$ of $G - \{r\}$ and outputs $T_{rl}, T_t, T_{ll}, T_{lh}$ and $T_{rh}$. We first have the following lemma.

**Lemma 3.2** Let $1 \leq k \leq m$, and let $T_{rl}^k$ be a graph induced by the right legs of all vertices in $G_k - \{r_b\}$. Then $T_{rl}^k$ is a spanning tree of $G_k$.

**Proof** We prove the claim by induction on $k$.

Clearly the claim holds for $k = 1$.

We assume that $1 \leq k \leq m - 1$ and $T_{rl}^k$ is a spanning tree of $G_k$, and we shall prove that $T_{rl}^{k+1}$ is a spanning tree of $G_{k+1}$. There are the following three cases to consider.

**Case 1:** $k \leq m-2$ and $W_{k+1}$ satisfies Condition (a) of (co3).

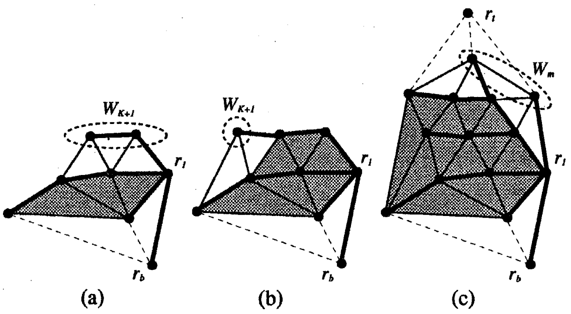


図 5: The three cases for Lemma 3.2.

**Case 2:** $k \leq m-2$ and $W_{k+1}$ satisfies Condition (b) of (co3).

**Case 3:** $k = m - 1$.

For each case $T_{rl}^{k+1}$ is a spanning tree of $G_{k+1}$ as shown in Fig. 5; (a) for Case 1; (b) for Case 2; and (c) for Case 3. Q.E.D.

We then have the following lemma.

**Lemma 3.3** $T_{rl}, T_t, T_{ll}, T_{lh}$ and $T_{rh}$ are spanning trees of $G$.

**Proof** By Lemma 3.2, $T_{rl}^m$ is a spanning tree of $G_m$, and hence $T_{rl}$ in which $r_b$ is regarded as $r$ is a spanning tree of $G$.

Similarly $T_t, T_{ll}, T_{lh}$ and $T_{rh}$ are spanning trees of $G$. Q.E.D.

Let $v$ be any vertex in $V - \{r\}$, and let $P_{rl}, P_t, P_{ll}, P_{lh}$ and $P_{rh}$ be the paths connecting $r$ and $v$ in $T_{rl}, T_t, T_{ll}, T_{lh}$ and $T_{rh}$, respectively. For any vertex $u$ in $V - \{r\}$ we write $rank(u) = k$ if $u \in W_k$; $rank(r)$ is undefined. If an edge $(v, u)$ of $G'$ is either a leg or a tail of vertex $v$, and $(v, w)$ of $G'$ is a hand of $v$, then $rank(u) \leq rank(v) \leq rank(w)$, and additionally if $v \neq r_2$ then $rank(u) < rank(w)$. See Fig. 4. Now we have the following lemma.

**Lemma 3.4** Every pair of paths $P_1 \in \{P_{rl}, P_t, P_{ll}\}$ and $P_2 \in \{P_{lh}, P_{rh}\}$ are internally disjoint.

**Proof** We prove only that $P_{rl}$ and $P_{rh}$ are internally disjoint. Proofs for the other pairs are similar. If $v = r_1$ then $P_{rl} = (v, r)$. If $v = r_{d(r)}$ then $P_{rh} = (v, r)$. If $v = r_2$ then $P_{rl} = (v, r_1, r)$

and $P_{rh} = (v, u_{d(r_2)-2}, \cdots)$. Therefor $P_{rl}$ and $P_{rh}$ are internally disjoint if $v$ is $r_1$, $r_2$ or $r_{d(r)}$. Thus we may assume that $v \neq r_1, r_2, r_{d(r)}$. Let $P_{rl} = (v, v_1, v_2, \cdots, v_l, r)$, then $v_l = r_1$. Let $P_{rh} = (v, u_1, u_2, \cdots, u_{l'}, r)$, then $u_{l'} = r_{d(r)}$. The definition of a right leg implies that $rank(v) \geq rank(v_1) \geq rank(v_2) \geq \cdots \geq rank(v_l)$, and the definition of a right hand implies that $rank(v) \leq rank(u_1) \leq rank(u_2) \leq \cdots \leq rank(u_{l'})$. Thus $rank(v_l) \leq \cdots \leq rank(v_2) \leq rank(v_1) \leq rank(v) \leq rank(u_1) \leq rank(u_2) \leq \cdots \leq rank(u_{l'})$. We furthermore have $rank(v_1) < rank(u_1)$ since $v \neq r_2$. Therefore $P_{rl}$ and $P_{rh}$ are internally disjoint. Q.E.D.

If $rl(v) = (v, u)$ then we say $(v, u)$ is an *incoming right leg* of $u$. Similarly, if $t(v) = (v, u)$ then $(v, u)$ is an *incoming tail* of u, and if $ll(v) = (v, u)$ then $(v, u)$ is an *incoming left leg* of $u$.

We have the following lemma.

**Lemma 3.5** Let $u \in V - \{r\}$, $ll(u) = (u, u')$, $rl(u) = (u, u'')$, and $N(u) = \{v_0, v_1, \cdots, v_{d(u)-1}\}$. One may assume that $u' = v_0$ and $u'' = v_z$ for some $z$, $3 \leq z \leq d(u) - 2$. Then all incoming right legs of $u$ appear consecutively around $u$. Also all incoming tails of $u$ appear consecutively around $u$, and all incoming left legs of $u$ appear consecutively around $u$. Furthermore $ll(u)$, the incoming right legs, incoming tails, incoming left legs and $rl(u)$ appear clockwise around $u$ in this order.

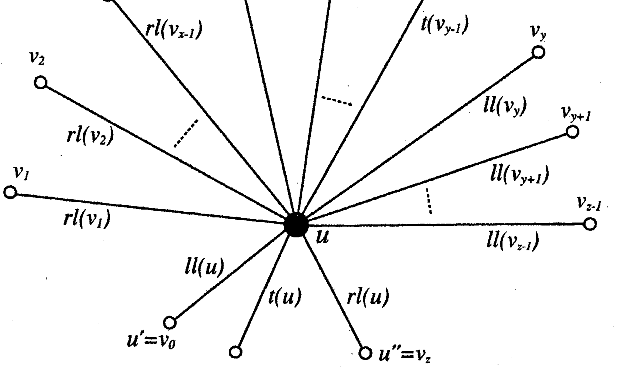


図 6: Illustration for Lemma 3.5.

**Proof** If $u = r_2$ then the claim is clearly holds. (In this case there is no incoming legs of $u$.) Thus we assume $u \neq r_2$.

If $(u_i, u)$ is the tail of $u_i \in W_k$ then $u \in C_o(G_{k-1})$ and $u \notin C_o(G_k)$. (See Fig. 4.) Thus if $t(u_i) = (u_i, u)$ and $t(u_j) = (u_j, u)$ then $\{u_i, u_j\} \in W_k$ for some $k$. Therefore all incoming tails of $u$ appear consecutively around $u$. (See Fig. 4.)

If $1 \leq i \leq z - 1$ and $rl(v_i) = (v_i, u)$, then $(v_{i-1}, u) \notin C_o(G_k)$, and either $t(u) = (u, v_{i-1})$, $rl(v_{i-1}) = (v_{i-1}, u)$ or $ll(u) = (u, v_{i-1})$ hold. (If $rank(v_i) = rank(u)$ then $t(u) = (u, v_{i-1})$. Otherwise assume $rank(v_i) = k$. Now edge $(v_{i-1}, u)$ is on $C_o(G_{k-1})$. If $rank(v_{i-1}) \leq rank(u)$ then $ll(u) = (u, v_{i-1})$. If $rank(v_{i-1}) \geq rank(u)$ then $rl(v_{i-1}) = (v_{i-1}, u)$. See Fig. 4.) Thus if u has an incoming right leg $e$ then the edge preceding $e$ around $u$ clockwise is either an incoming right leg of $u$, $t(u)$ or $ll(u)$. Since $t(u)$ and $ll(u)$ always appear consecutively around $u$, therefore all incoming right legs of $u$ appear consecutively around $u$ and $ll(u)$ precedes them. Similarly all incoming left legs of $u$ appears consecutively around $u$ and $rl(u)$ succeeds them. Thus the claim holds. Q.E.D.

Lemma 3.5 immediately implies the following lemma.

**Lemma 3.6** A pair of paths $P_1, P_2 \in \{P_{rl}, P_t, P_{ll}\}$ may cross at a vertex $u$, but do not share a vertex $u$ without crossing at $u$.

From the definitions of a left leg , a tail and a right leg one can immediately have the following lemma.

**Lemma 3.7** Let $1 \leq k \leq m$, $u \neq r_2$ and $u \in W_k$. Then $u$ is on $C_o(G_k)$. Let $u'$ be the succeeding vertex of $u$ on $C_o(G_k)$. Assume that the ordered set $N(u)$ starts with $u'$. Let $rl(u) = (u, v')$, $t(u) = (u, v'')$ and $ll(u) = (u, v''')$. Then $v'$, $v''$, $v'''$ appear in $N(u)$ in this order.

We then have the following lemma.

**Lemma 3.8** A pair of paths $P_1, P_2 \in \{P_{rl}, P_t, P_{ll}\}$ are internally disjoint. Also $P_{lh}, P_{rh}$ are internally disjoint.

**Proof** We prove only that $P_{rl}$ and $P_{ll}$ are internally disjoint. Proofs for the other cases are similar. Suppose for a contradiction that $P_{rl}$ and $P_{ll}$ share an intermediate vertex. Let $w$ be the intermediate vertex that is shared by $P_{rl}$ and $P_{ll}$ and appear last on the path $P_{rl}$ going from $r$ to $v$. Now $w \neq r_2$ because $r_2$ has degree one in both $T_{rl}$ and $T_{ll}$. Then $P_{rl}$ and $P_{ll}$ cross at $w$ by Lemma 3.6. However, the claim in Lemma 3.7 holds both for $k = rank(v)$ and $u = v$ and for $k = rank(w)$ and $u = w$, and hence $P_{rl}$ and $P_{ll}$ do not cross at $w$, a contradiction. Q.E.D.

By Lemmas 3.4 and 3.8 we have the following lemma.

**Lemma 3.9** $T_{rl}, T_t, T_{ll}, T_{lh}$ and $T_{rh}$ are five independent spanning trees of $G$ rooted at $r$.

Clearly the running time of Algorithm Five-Trees is $O(n)$. Thus we have the following theorem.

**Theorem 3.10** Five independent spanning trees of any 5-connected maximal planar graph rooted at any designated vertex can be found in linear time.

## 4 Proof of Lemma 3.1

In this section we give an algorithm to find a 5-canonical decomposition. Then we show it runs in linear time. First we need some definitions.

Let $G = (V, E)$ be a 5-connected maximal plane graph, let $r$ be a designated vertex on $C_o(G)$, and let $H$ be a triconnected plane subgraph of $G''$ such that $r_b \in C_o(H)$. Let $C_o(H) = (r_b = w_1, w_2, \cdots, w_l)$.

A set of edges $(v_1, u), (v_2, u), \cdots, (v_h, u)$ in $H$ is called a *fan with center* $u$ if (1) $u \notin C_o(H)$, (2) the neighbors of $u$ on $C_o(H)$ are $v_1, v_2, \cdots, v_h$, called *leaves*, and they appear in $C_o(H)$ clockwise in this order, and (3) either $h = 2$ and $H$ does not have $edge(v_1, v_2)$, or $h \geq 3$. Assume a set of edges $(v_1, u), (v_2, u), \cdots, (v_h, u)$ is a fan $F$ with center $u$ Now, for $1 \leq i \leq h-1$, $v_i = w_a$ and $v_{i+1} = w_b$ hold for some $a$, $b$ such that $1 \leq a < b \leq l$, and let $C_i$ be the cycle consisting of the subpath $(w_a, w_{a+1}, \cdots, w_b)$ of $C_o(H)$ and two edges $(w_b, u), (u, w_a)$. Each plane subgraph $F_i$ of $H$ inside $C_i$ (including $C_i$) is called a *piece* of $F$. $F_i$ is called an *empty piece* if $a + 1 = b$. If $F_i$ is an empty piece then $C_i$ is a triangle face of $H$. (Since G is 5-connected, if $a + 1 = b$ then $F_i$ has no vertex in the proper inside.) Note that by the definition if a fan has exactly two leaves then it has exactly one piece and the piece is not empty. Also note that $F$ has exactly $h - 1$ pieces, and if $v_1 \neq r_b$ then none of pieces of $F$ contains $r_b$. If none of pieces of $F$ contains a distinct fan, then $F$ is a *minimal* fan.

A *cut-set* is a set of vertices whose removal results in a disconnected graph. Since $G$ is 5-connected and maximal planar, every cut-set of $H$ consisting of three vertices has (1) exactly one vertex not in $C_o(H)$ and (2) exactly two vertices in $C_o(H)$. Thus each cut-set of $H$ consisting of three vertices corresponds to a center of a fan and its two leaves.

We have the following lemmas.

**Lemma 4.1** If a vertex $v \in C_o(H)$ is contained in none of fans of $H$ (Note that, however, $v$ may be contained in a piece of a fan.), then $H - \{v\}$ is triconnected, where $H - \{v\}$ is the plane subgraph of $H$ obtained from $H$ by deleting $v$ and all edges incident to $v$.

**Lemma 4.2** If all pieces of a fan $F = (v_1, u), (v_2, u), \cdots, (v_h, u)$ of $H$ is empty (Now $d(v_1) \geq 4$, $d(v_h) \geq 4$ and, for $j = 2, 3, \cdots, h-1$, $d(v_j) = 3$.) and $u \neq r_2$, then $H - \{v_2, v_3, \cdots, v_{h-1}\}$ is triconnected, where $H - \{v_2, v_3, \cdots, v_{h-1}\}$ is a plane subgraph of $H$ obtained from $H$ by deleting $v_2, v_3, \cdots, v_{h-1}$ and all edges incident to them.

Now we give our algorithm to find a 5-canonical decomposition.

First, by Condition (co1) we can find $W_m$. Now $\overline{G_{m-1}}$ is biconnected since $\overline{G_{m-1}}$ is a triangle cycle. Since $G = (V, E)$ is 5-connected, the vertex set $V - \{r\}$ induces a 4-connected graph $G'$. And $G_m$ is obtained from $G'$ by adding a new vertex $r_b$ adjacent three vertices of $G'$. Now $G_m$ is triconnected since a graph obtained from a $k$-connected graph $G$ by adding a new vertex adjacent $k$ vertices of $G$ is also $k$-connected [W96, p145]. Also $G_{m-1}$ is triconnected, since otherwise $G_{m-1}$ has a cut-set $S$ with two or less vertices and then $S \bigcup W_m$ is a cut-set of $G$ with four or less vertices, a contradiction. Thus for $k = m-1$ and $m$, $G_k$ is triconnected, and for $k = m-1$, $\overline{G_k}$ is biconnected. Clearly $r_1, r_2, r_3 \notin W_m$.

Then, inductively assume that we have chosen $W_m, W_{m-1}, \cdots, W_{i+1}$ such that for each $k = i, i+1, \cdots, m$, $G_k$ is triconnected, and for each $k = i, i+1, \cdots, m-1$, $\overline{G_k}$ is biconnected, $r_1, r_2, r_3 \notin W_m \bigcup W_{m-1} \bigcup \cdots \bigcup W_{i+1}$ and each $W_k$, $k = i+1, i+2, \cdots, m$, satisfies either (co1) or (co3). Now we can choose $W_i$ as follows. We have two cases. If $G_i$ has exactly one vertices in the proper inside of $G_i$ then it is $r_2$ and we have done by setting all vertices in $G_i$ except $r_b$ as $W_1$. Otherwise we can find $W_i \subseteq V - W_m \bigcup W_{m-1} \bigcup \cdots \bigcup W_{i+1}$ such that (1) $G_{i-1}$ is triconnected, (2) $\overline{G_{i-1}}$ is biconnected, (3) $r_1, r_2, r_3 \notin W_i$, (4) $W_i$ satisfies (co3), as follows.

Let $F = (v_1, u), (v_2, u), \cdots, (v_h, u)$ be a minimal fan of $G_i$. Note that $G_i$ always has a fan $(r_b, r_2), (r_3, r_2), \cdots, (r_1, r_2)$ with center $r_2$ implies $G_i$ always has a fan.

If every piece of $F$ is empty then $F$ has three or more leaves, and we can set $W_i = \{v_2, v_3, \cdots, v_{h-1}\}$. Now if $h \geq 4$ then $W_i$ satisfies (a) of (co3) and $G_{i-1}$ is triconnected by Lemma 4.2, and $\overline{G_{i-1}}$ is biconnected since each vertex in $W_i$ has degree exactly three in $G_i$ means each vertex in $W_i$ has two or more neighbors in $\overline{G_i}$. Similarly if $h = 3$ then $W_i$ satisfies (b) of (co3), and $G_{i-1}$ is triconnected by Lemma 4.2, and $\overline{G_{i-1}}$ is biconnected as above.

Otherwise, let $F'$ be a non-empty piece of $F$. Now $F'$ has four or more vertices on $C_o(G_i)$ since otherwise $G$ has a cut-set with four or less vertices, a contradiction. Now there exists at least one vertex of $F'$ on $C_o(G_i)$ such that (1) it is not a leaf of $F$, and (2) it has two or more neighbors in $\overline{G_i}$. (Since otherwise each vertices of $F'$ on $C_o(G_i)$ except the two leaves $w_a, w_b$ of $F$ has at most one neighbor in $\overline{G_i}$, and for $G$ is maximal planar each neighbor in $\overline{G_i}$ is a common vertices, say $x$, and $\{u, w_a, w_b, x\}$ forms a cut-set, a contradiction.) Thus we can find $W_i$ satisfying (b) of (co3). Now $G_{i-1}$ is triconnected by Lemma 4.1, and $\overline{G_{i-1}}$ is biconnected.

Thus we can find a 5-canonical decomposition. By maintaining a data-structure to keep fans and the number of neighbors in $\overline{G_i}$ for each vertex, the algorithm runs in linear time.

## 5 Conclusion

In this paper we give a linear-time algorithm to find $k$ independent spanning trees of a $k$-connected maximal planar graph rooted at any designated vertex. It is remained as future work to find a linear-time algorithm for planar graphs, which are not always maximal planar.

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